Artin groups of spherical type up to commensurability

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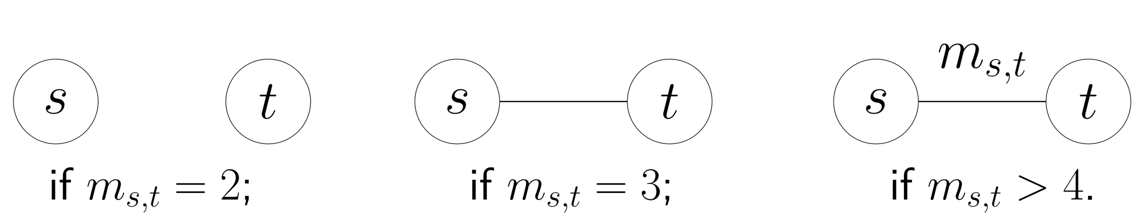
Abstract. *Let A and A' be two Artin groups of spherical type, and let $A_1, \dots, A_p$ (resp. $A'_1, \dots, A'_q$) be the irreducible components of A (resp. A'). We show that A and A' are commensurable if and only if $p = q$ and, up to permutation of the indices, A_i and A'_i are commensurable for every i. We prove that, if two Artin groups of spherical type are commensurable, then they have the same rank. For a fixed n, we give a complete classification of the irreducible Artin groups of rank n that are commensurable with the group of type A_n. Note that there will remain 6 pairs of groups to compare to get the complete classification of Artin groups of spherical type up to commensurability. This is a joint work with Luis Paris.*

Coxeter graphs and Artin groups

Let S be a finite set and $M = (m_{s,t})_{s,t \in S}$ a symmetric matrix with $m_{s,s} = 1$ and $m_{s,t} \in \{2, \dots, \infty\}$ for $s \neq t$. The Coxeter graph Γ_M is the graph with set of vertices S and



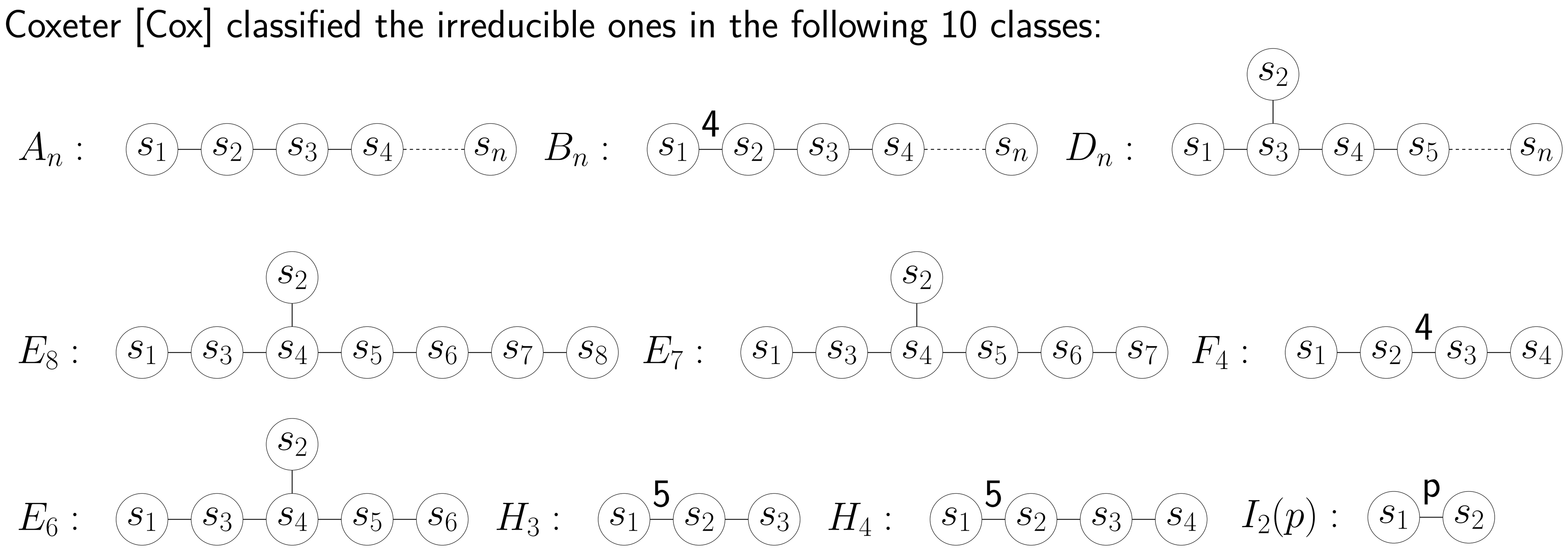
The Artin group associated to Γ_M, denoted by $A[\Gamma_M]$, is the group with the following presentation

$$A = \langle \Sigma \mid \underbrace{sts\dots}_{m_{s,t}\text{ elements}} = \underbrace{tst\dots}_{m_{s,t}\text{ elements}} \;\forall s,t \in S,\ s \neq t,\ m_{s,t} \neq \infty \rangle.$$

A is irreducible if its Coxeter graph is connected.

Artin groups of spherical type

If, by adding $s^2 = 1$ to the relations of A, we obtain a finite group, we say that A has spherical type. Coxeter [Cox] classified the irreducible ones in the following 10 classes:



Definition Two groups G_1 and G_2 are commensurable if there are two finite index subgroups H_1 of G_1 and H_2 of G_2 such that H_1 is isomorphic to H_2.

Aim Classifying Artin groups of spherical type up to commensurability.

- **Reducible case:** The following theorem shows that this case entirely depends on the irreducible case.

Theorem [CP]

Let $A[\Gamma]$, $A[\Omega]$ be two Artin–Tits groups of spherical type with Coxeter graphs Γ and Ω having decompositions

$$\Gamma = \Gamma_1 \sqcup \Gamma_2 \sqcup \dots \sqcup \Gamma_p, \quad \Omega = \Omega_1 \sqcup \Omega_2 \sqcup \dots \sqcup \Omega_p \qquad (\Gamma_i \text{ and } \Omega_i \text{ are connected components.})$$

$A[\Gamma]$ and $A[\Omega]$ are commensurable if and only if $A[\Gamma_i]$ and $A[\Omega_i]$ are commensurable (up to permutation).

- **Irreducible case:** The following result allows us to significantly reduce the pairs of potential commensurable groups.

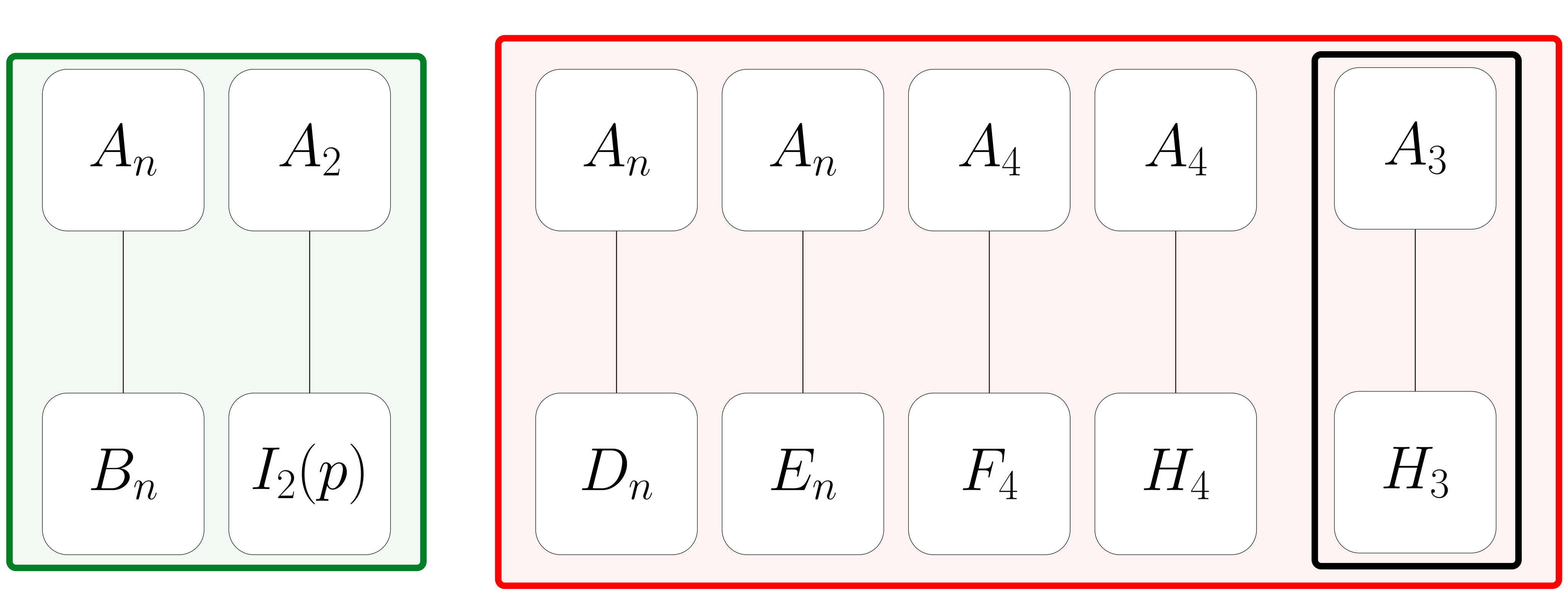
Proposition [CP]

If $A[\Gamma]$ and $A[\Omega]$ are commensurable Artin groups of spherical type, then the Coxeter graphs Γ and Ω have the **same number of vertices**.

Definition We say that an element α in a group G is a generalized torsion element if there are $p \geq 1$ and $\beta_1, \dots, \beta_p \in G$ such that $(\beta_1 \alpha \beta_1^{-1})(\beta_2 \alpha \beta_2^{-1}) \cdots (\beta_p \alpha \beta_p^{-1}) = 1$. We say that G has generalized torsion if it contains a non-trivial generalized torsion element.

Partial classification

These pairs are commensurable.
The proofs are direct and short.



The groups corresponding to these pairs are **not commensurable**. To prove it we use the following result:

Lemma [CP]

If the kernel of every homomorphism $\varphi : A[\Gamma]/Z(A[\Gamma]) \to \mathfrak{S}_{n+2} \times \{\pm 1\}$ has generalized torsion*, then $A[\Gamma]$ and $A[A_n]$ are not commensurable. *$\mathfrak{S}_n$ is the symmetric group with n elements.

Using a software, we check that in every case (but for the pair (A_3, H_3)) we can construct generalized torsion elements in $\mathrm{Ker}(\varphi)$, for every homomorphism φ.

For the special case of (A_3, H_3), we prove that there is no injective homomorphism from $A[H_3]/A[H_3]$ to $\mathcal{M}^*(\Sigma_{0,0}, 5)$, and that this implies that the groups are not commensurable.

Definition Let $\Sigma = \Sigma_{g,b}$ be the orientable surface of genus g The extended mapping class group of the pair (Σ, n), denoted by $\mathcal{M}^*(\Sigma, n)$, is the group of isotopy classes of homeomorphisms $h : \Sigma \to \Sigma$ that fix the boundary of Σ pointwise and preserve a set of n different points inside Σ. *(More in [FM]).*

Open problem To finish the classification we need to know if the groups of these pairs are commensurable: (D_6, E_6), (D_7, E_7), (D_8, E_8), (D_4, F_4), (D_4, H_4), (F_4, H_4).

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