The complex of irreducible parabolic subgroups

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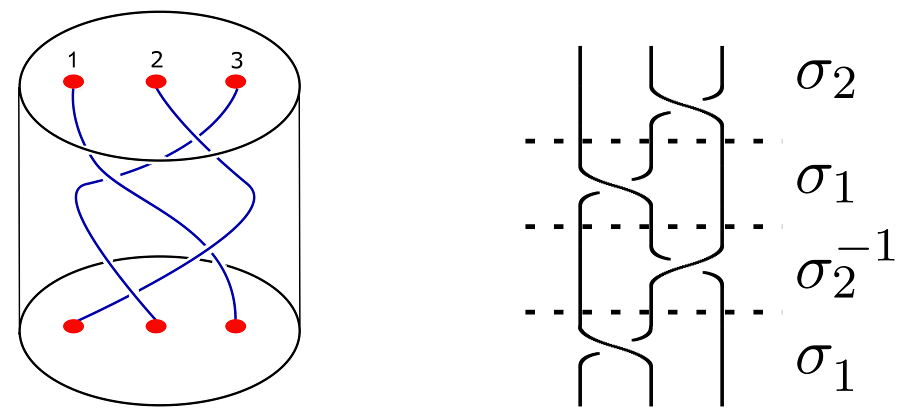
Abstract. *Artin groups of spherical type are a algebraic generalisation of braid groups and share many properties with them. However, some properties of the braid group are proved using topological tools that does not apply for Artin groups, since a braid group can be seen as a mapping class group of a n-punctured disc, $\mathcal{D}_n$. Most of these techniques use the action of the braid group on the curve complex of $\mathcal{D}_n$. In order to extend this kind of results to Artin groups of spherical type, we present the new complex of proper irreducible parabolic subgroups of Artin groups of spherical type. This is a joint work with Volker Gebhardt, Juan González-Meneses and Bert Wiest.*

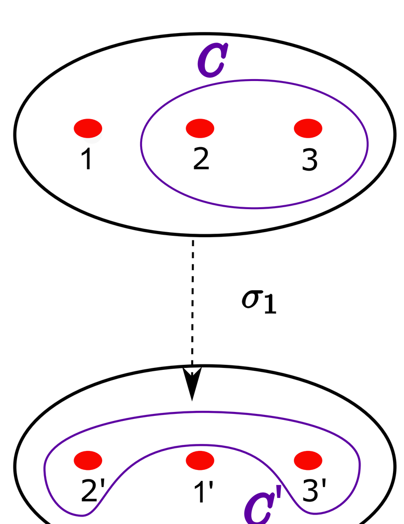
The braid group

A braid with n strands can be seen as a collection of n disjoint paths in a cylinder, defined up to isotopy, joining n points at the top with n points at the bottom, running monotonically in the vertical direction.



The braid group is generated by $\sigma_1, \dots, \sigma_{n-1}$, where each σ_i represents a crossing between the strands in position i and $i+1$ with a fixed orientation [Artin]. In the picture $\alpha = \sigma_2\sigma_1\sigma_2^{-1}\sigma_1$ is represented. The standard presentation for the braid group in n strands is:

$$\mathcal{B}_n = \left\langle \sigma_1, \dots, \sigma_{n-1} \middle| \begin{matrix} \sigma_i\sigma_j = \sigma_j\sigma_i, & |i-j|>1 \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j, & |i-j|=1 \end{matrix} \right\rangle.$$



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A braid acts on the set of isotopy classes of simple closed curves on $\mathcal{D}_n$.

Actually $\mathcal{B}_n$ is the mapping class group of $\mathcal{D}_n$.

For example, the braid σ_1 turns the curve C on the left into C'.

Curve complex of $\mathcal{D}_n$

The curve complex $\mathcal{C}(S)$ of a surface S [FM, Ch. 4] is a simplicial flag complex such that

- The set of vertices is the set of isotopy classes of non-degenerated simple closed curves.
- There is an edge between two vertices if the corresponding curves can be isotoped to be disjoint.

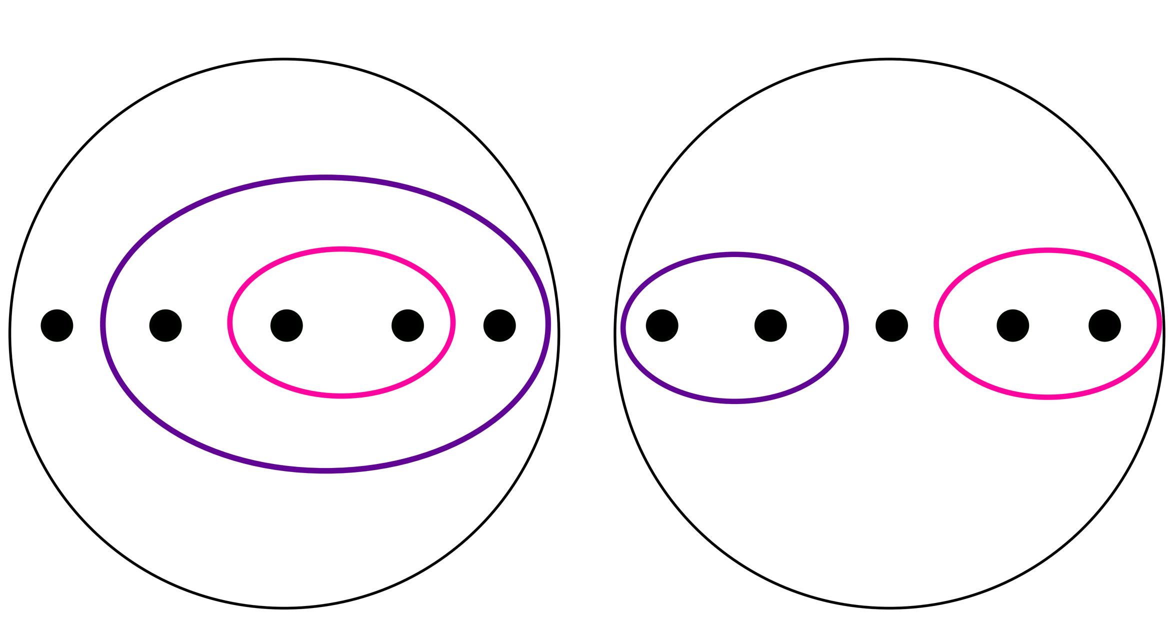

Figure 1: The two adjacency situations in $\mathcal{C}(\mathcal{D}_n)$

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Artin group A of spherical type

Let S be a finite set and $M = (m_{i,j})_{i,j\in S}$ a symmetric matrix with $m_{i,i} = 1$ and $m_{i,j} \in \{2, \dots, \infty\}$ for $i \neq j$. Let $\Sigma = \{\sigma_i \,|\, i \in S\}$. The Artin system [Tit] associated to M is (A, Σ), where A is a group with the following presentation

$$A = \langle \Sigma \mid \underbrace{\sigma_i\sigma_j\sigma_i \dots}_{m_{i,j} \text{ elements}} = \underbrace{\sigma_j\sigma_i\sigma_j \dots}_{m_{i,j} \text{ elements}} \forall i,j \in S,\, i \neq j,\, m_{i,j} \neq \infty \rangle.$$

By adding the relation $\sigma_i^2 = 1$, the associated Coxeter group is obtained. If the Coxeter group is finite, then A is said to be an Artin group of spherical type. The main example is the braid group.

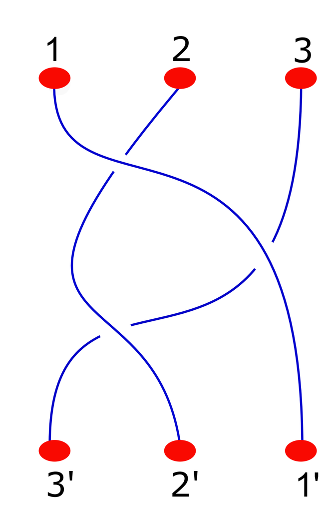
Parabolic subgroup P of A

A subgroup $A_X \subset A$ generated by a subset X of Σ is a standard parabolic subgroup.

If we conjugate A_X by $\alpha \in A$ we obtain a parabolic subgroup, denoted by $P = \alpha^{-1}A_X\alpha$.

z_P: the generator of $Z(P)$

For each A of spherical type it exists an element Δ such that Δ^e generates the center of A, for $e = 1$ or $e = 2$. For the braid group, Δ is a half-twist of the trivial braid (on the right). For A_X, this element is denoted Δ_X.



For a general $P = \alpha^{-1}A_X\alpha$ the generator of the center is $z_P = \alpha^{-1}\Delta_X^e\alpha$ [Par].

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Parabolic subgroups and simple closed curves

A standard curve can be associated to the subgroup generated by the set of generators involving only its enclosed punctures. This is why parabolic subgroups are an algebraic analogue of simple closed curves.

The curve in Figure 2 corresponds to $A_{\{\sigma_2,\sigma_3\}}$. If the braid α acts on D_n, then the corresponding parabolic subgroup is $\alpha^{-1}A_{\{\sigma_2,\sigma_3\}}\alpha$.

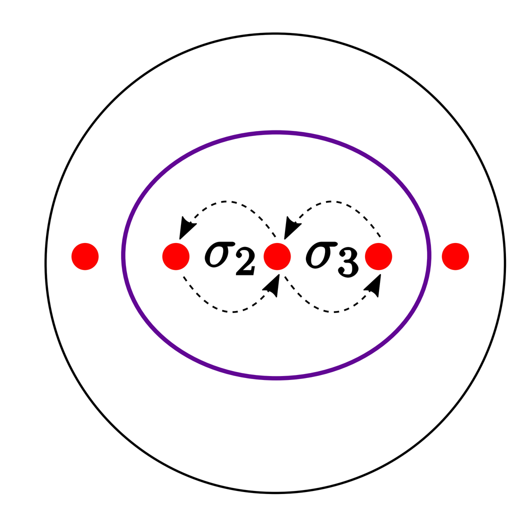

Figure 2: $A_{\{\sigma_2,\sigma_3\}}$

Complex of irreducible parabolic subgroups of A

This complex [CGGW], denoted by $\mathcal{P}$, is a simplicial flag complex such that:

- **Set of vertices**: Proper irreducible parabolic subgroups of A, where
  - P is proper if $P \neq \{1\}, A$.
  - $P = \alpha^{-1}A_X\alpha$ is irreducible if A_X cannot be decomposed as a non-trivial direct product of standard parabolic subgroups.
- **Adjacency condition**: There is an edge between two vertices if the corresponding parabolic subgroups P and Q are such that $z_Pz_Q = z_Qz_P$.

Why this adjacency condition?

Notice that, if P, Q are vertices of $\mathcal{P}$, the two adjacency situations in Figure 1 are equivalent to

1. $P \subset Q$ or $Q \subset P$.
2. $P \cap Q = \{1\}$ and $xy = yx$, $\forall x \in P, \forall y \in Q$.

Theorem [CGGW]

We have adjacency conditions 1 or 2 above if and only if $z_Pz_Q = z_Qz_P$.

Remark. *We have been able to define $\mathcal{P}$ thanks to the main result of [CGGW], which was a basic question unknown for a long time:*

Theorem [CGGW]

The intersection of two parabolic subgroups is a parabolic subgroup.

Corollary [CGGW]

The set of parabolic subgroups is a lattice with respect to the inclusion.

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