Minimal standardizer of a parabolic subgroup of an Artin-Tits group

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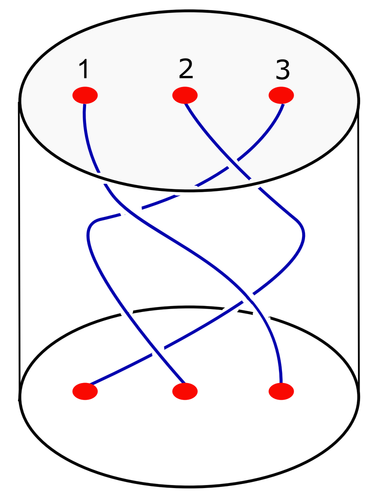
Abstract. *The minimal standardizer of a curve is the minimal positive braid that transforms it into a round one. We give an algorithm to compute it in a geometrical way, using the concept of bending point. Then, we generalize this problem algebraically to parabolic subgroups of Artin-Tits groups of spherical type and we show that, to compute the minimal standardizer of a parabolic subgroup, it suffices to compute the pn-normal form of the generator of its center.*

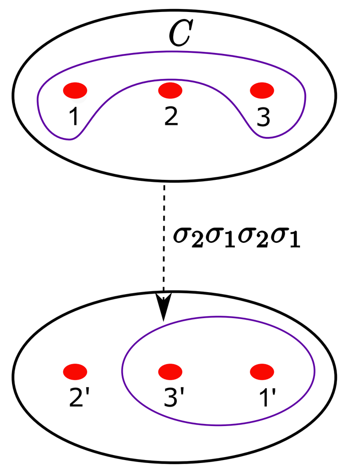
Definition of braid

A braid with n strands can be seen as a collection of n disjoint paths in a cylinder, defined up to isotopy, joining n points at the top with n points at the bottom, running monotonically in the vertical direction.

The braid group is generated by $\sigma_1, \dots, \sigma_{n-1}$, where each σ_i represents a crossing between the strands in position i and $i+1$ with a fixed orientation [1]. In the picture $\alpha = \sigma_2\sigma_1\sigma_2\sigma_1$ is represented. This braid is positive, as all generators appear with positive exponent.



How to standardize a curve on D_n?



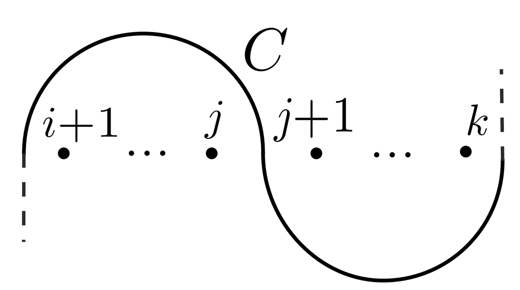
A braid can be also seen has an automorphism of the disk with n punctures, D_n, which fixes its border. A braid acts on the set of isotopy classes of simple closed curves on D_n.

For example, α turns the following curve C on the left into a round one (also called standard). We will say that α is a standardizer of C. The set of all positive standardizers of C is denoted $St(C)$.

LEE & LEE [5]: There is a unique minimal element on $St(C)$
C. : Let's compute it!

Bending point

The curve C has a bending point at j if a part of the curve is as in the picture below (up to deformation). If a curve is not standard, then it has a bending point. [3]



Algorithm for curves (C.)

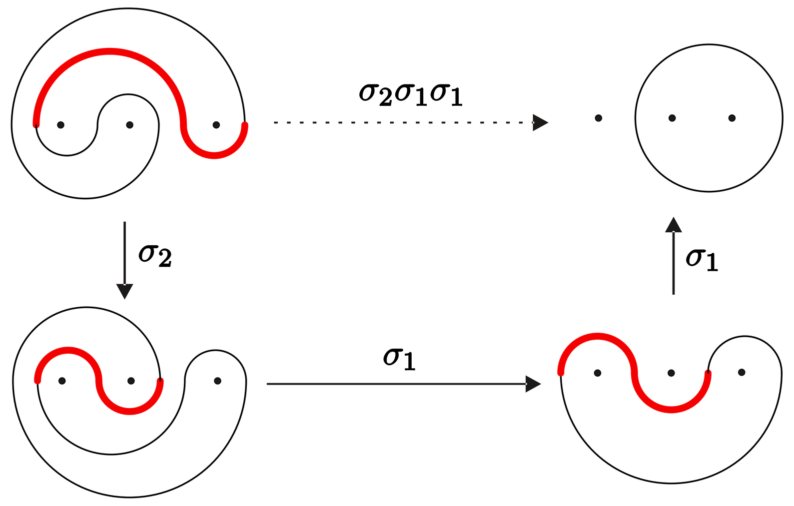
INPUT: A curve C.

1. Set $\alpha = 1$;
2. While C has a bending point at $j = 1, \dots, n-1$ do:
   2.1 $\alpha = \alpha\sigma_j$;
   2.2 $C = C^{\sigma_j}$ (Apply σ_j to C);

OUTPUT: α is the minimal standardizer of C.

Example 1

In order to standardize the curve on the top left side of the picture, the algorithm realises that there is a bending point at 2, and then applies σ_2. Iterating, we obtain that the minimal standardizer of the curve is $\sigma_2\sigma_1\sigma_1$



Artin-Tits group of spherical type

Let S be a finite set and $M = (m_{i,j})_{i,j\in S}$ a symmetric matrix with $m_{i,i} = 1$ and $m_{i,j} \in \{2, \dots, \infty\}$ for $i \neq j$. Let $\Sigma = \{\sigma_i \mid i \in S\}$. The Artin-Tits system associated to M is (A, Σ), where A is a group with the following presentation

$$A = \langle \Sigma \mid \underbrace{\sigma_i\sigma_j\sigma_i\dots}_{m_{i,j}\text{ elements}} = \underbrace{\sigma_j\sigma_i\sigma_j\dots}_{m_{i,j}\text{ elements}} \forall i,j \in S,\ i \neq j,\ m_{i,j} \neq \infty\rangle.$$

By adding the relation $\sigma_i^2 = 1$, the associated Coxeter group is obtained. If the Coxeter group is finite, then A is said to be of spherical type. The main example is the braid group.

Parabolic subgroup of A

A subgroup $A_X \subset A$ generated by a subset X of Σ is a standard parabolic subgroup.

If we conjugate A_X by $\alpha \in A$ we obtain a parabolic subgroup, denoted by $P = (A_X, \alpha) = \alpha A_X \alpha^{-1}$.

How to standardize a parabolic subgroup?

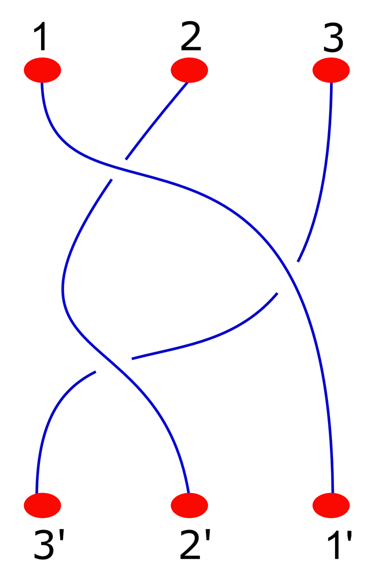
Keep in mind that A is a Garside group [2]. This implies that A has a submonoid of positive elements, A^+, and a partial order $\preccurlyeq$ (resp. $\succcurlyeq$), defined by $a \preccurlyeq b \Leftrightarrow a^{-1}b \in A^+$ (resp. $a \succcurlyeq b \Leftrightarrow ab^{-1} \in A^+$) such that for all $a, b \in A$ it exists a unique gcd $a \wedge b$ (resp. $a \wedge^\urcorner b$) and a unique lcm $a \vee b$ (resp. $a \vee^\urcorner b$). For a parabolic subgroup $P = (A_X, \alpha)$, we want to compute the $\preccurlyeq$-minimal $\beta \in A^+$ such that $\beta^{-1}P\beta$ is standard.

Garside element Δ

For each A of spherical type it exists an element

$$\Delta = \bigvee_{i\in S} \sigma_i$$

such that Δ^e generates the center of A, for $e = 1$ or $e = 2$. For the braid group, Δ is a half-twist of the trivial braid (on the right).



For A_X, the element is denoted Δ_X. For $P = (A_X, \alpha)$ the generator of the center is $\Delta^e_{X,\alpha} = \alpha\Delta^e_X\alpha^{-1}$ [6].

pn-normal form

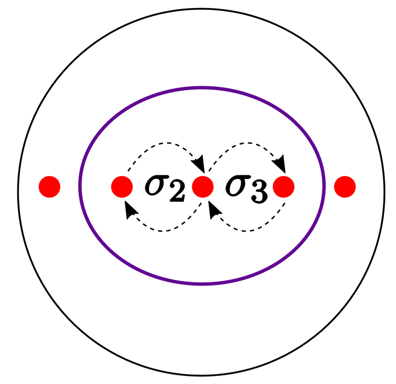
Let $a, b \in \mathcal{P}$, we say that $x = ab^{-1}$ is in pn-normal form if $a \wedge^\urcorner b = 1$. [4]

Theorem (C.)

Let $P = (A_X, \alpha)$ be a parabolic subgroup. If $\Delta^e_{X,\alpha} = ab^{-1}$ is in pn-normal form, then b is the minimal standardizer of P.

What has to do a curve with a parabolic subgroup?

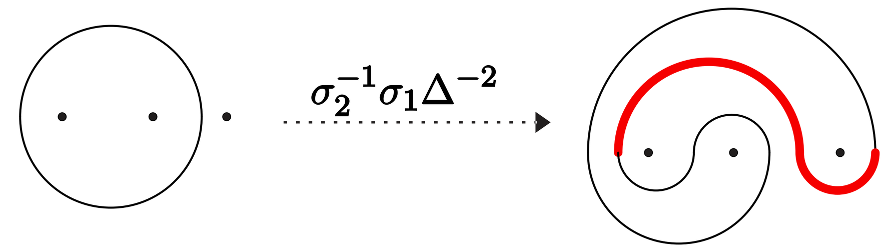
A standard curve can be associated to the subgroup generated by the set of generators involving only its enclosed punctures.



This is why parabolic subgroups are a generalization of simple closed curves. The curve in the picture corresponds to $A_{\{\sigma_2,\sigma_3\}}$. If the braid α acts on D_n, then the corresponding parabolic subgroup is $(A_{\{\sigma_2,\sigma_3\}}, \alpha^{-1})$.

Example 2:

Let us treat algebraically the same case of Example 1. The input of the new algorithm has to be the conjugate of a standard parabolic subgroup. A possible input could be $P = (A_{\{\sigma_1\}}, \sigma_2^{-1}\sigma_1\Delta^{-2})$.



Example of algorithm for parabolic subgroups:

1. $\Delta_{\{\sigma_1\}} = \sigma_1$ generates $Z(A_{\{\sigma_1\}})$.
2. $\Delta_{\{\sigma_1\},\sigma_2^{-1}\sigma_1\Delta^{-2}} = \sigma_1^{-1}\sigma_2\sigma_1\sigma_2^{-1}\sigma_1$.
3. The pn-normal form of $\Delta_{\{\sigma_1\},\sigma_2^{-1}\sigma_1\Delta^{-2}}$ is $(\sigma_2\sigma_1\sigma_1\sigma_2) \cdot (\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1})$.
4. Hence, the minimal standardizer is $\sigma_2\sigma_1\sigma_1$.

Complexity of algorithms

Curves: $O(n^2 m \log(m))$
m: number of intersections of the curve with the real axis.

Parabolic subgroups: $O(\ell^2)$
ℓ: length of $\Delta_{X,\alpha}$.

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