# Parabolic subgroups of large-type Artin groups

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***Abstract.*** *We explain how to prove that the intersection of parabolic subgroups of large-type Artin groups is a parabolic subgroup. This is a joint work with Alexandre Martin.*

## Artin groups

Let $S$ be a finite set and $M = (m_{s,t})_{s,t\in S}$ a symmetric matrix with $m_{s,s} = 1$ and $m_{s,t} \in \{2, \dots, \infty\}$ for $s \neq t$. The *Coxeter graph* $\Gamma_M$ is the graph with set of vertices $S$ and


The Artin group associated to $\Gamma_M$, is the group with this presentation:

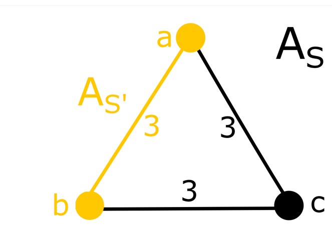
$$A = \langle S \mid \underbrace{sts\dots}_{m_{s,t}\text{ elements}} = \underbrace{tst\dots}_{m_{s,t}\text{ elements}} \ \forall s,t \in S,\ s \neq t,\ m_{s,t} \neq \infty\rangle.$$

$A$ has large type if all $m_{s,t} > 2$.

## Parabolic subgroups

A subgroup $A_{S'} \subset A$ generated by a subset $S'$ of $S$ is called a *standard parabolic subgroup*.

Standard parabolic subgroups are again Artin groups:




If we conjugate $A_{S'}$ by $\alpha \in A$ we obtain a parabolic subgroup, denoted by $P = \alpha^{-1}A_{S'}\alpha$.

**Open question** Is the intersection of parabolic subgroups a parabolic subgroup?

The parabolicity of the intersection of parabolic subgroups is a basic non-trivial question that had been only answered (on the positive) for the family of spherical Artin groups [CGGW] and for spherical parabolic subgroups of FC-type Artin groups [MW].

## Theorem [CMV]

**For the large case, the set of parabolic subgroups is stable under arbitrary intersections.**

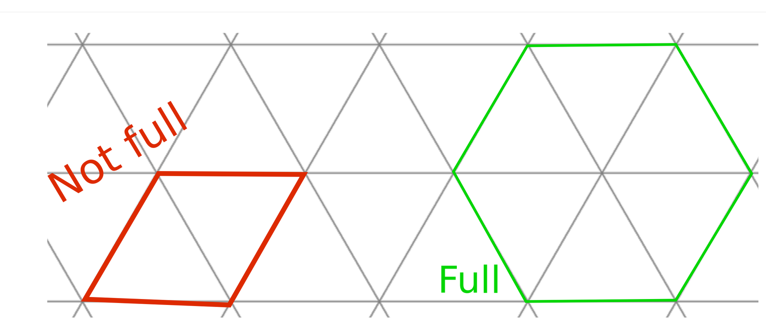
## Keys for the proof of the theorem

### Systolicity (JS)

Given a complex $X$, the systole of $X$ is

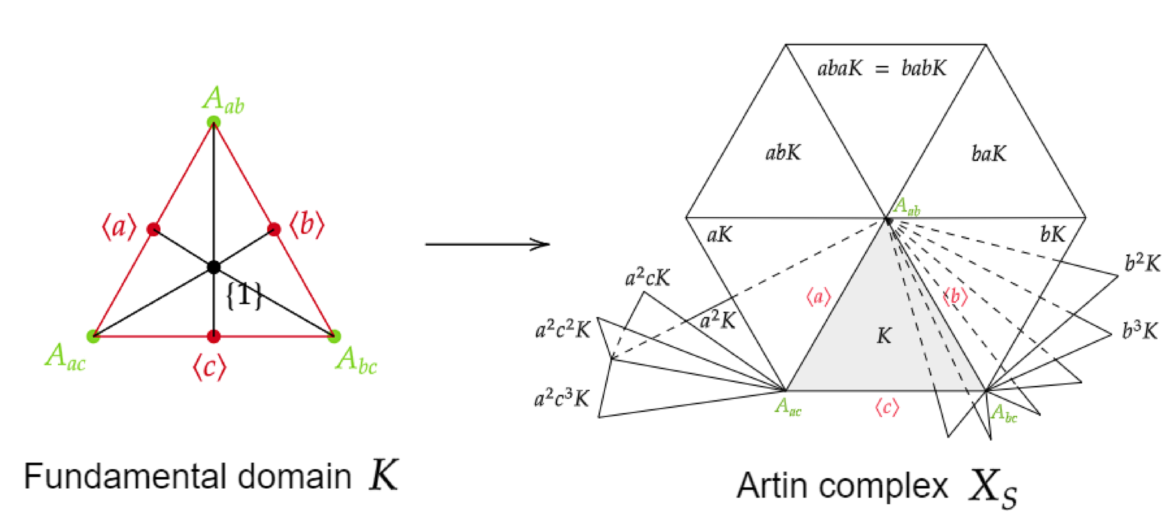
$\mathrm{Sys}(X) = \min\{|\gamma| : \gamma$ is a cycle in the 1-skeleton s.t. any subcomplex of $X$ with its vertices in $\gamma$ is also in $\gamma$.$\}$

A complex $X$ is systolic if it is connected, simply connected and the systole of every link $\geq 6$.

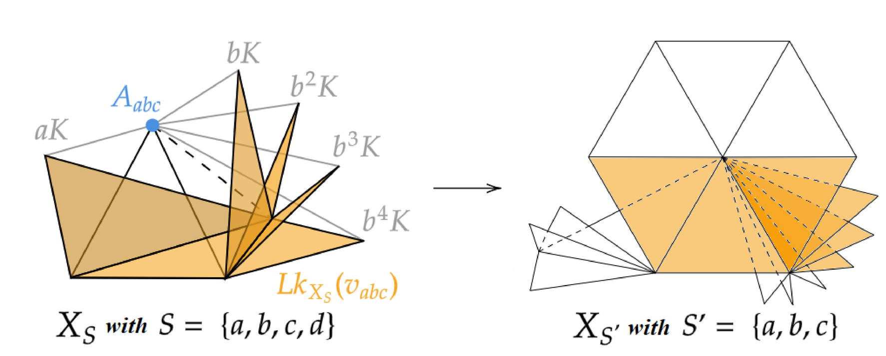



### Artin complex

The Artin complex $X_S$ associated with an Artin group $A_S$ is the geometric realisation of the poset of left cosets of standard parabolic subgroups of $A_S$.




**Lemma**: The link of a simplex in the Artin complex of an Artin group is isomorphic to an Artin complex of a smaller Artin group.

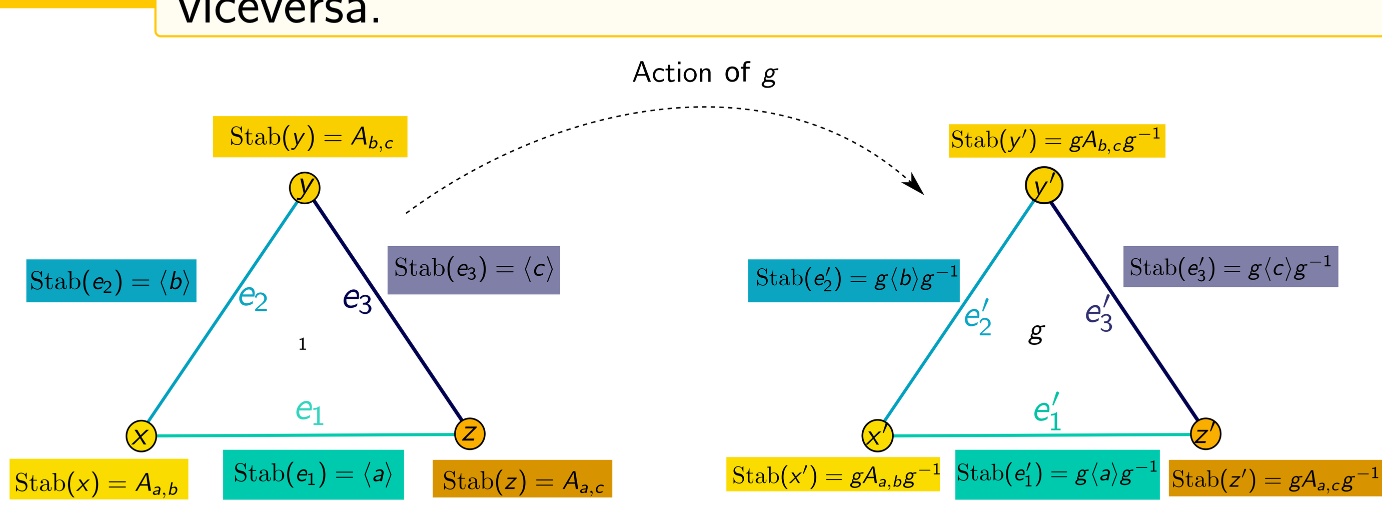



Since the systolicity of a complex does not depend on its systole but on the systole of links, the previous lemma allows us to apply inductive arguments to proof the following:

## Proposition [CMV]

**For large-type, the Artin complex is systolic.**

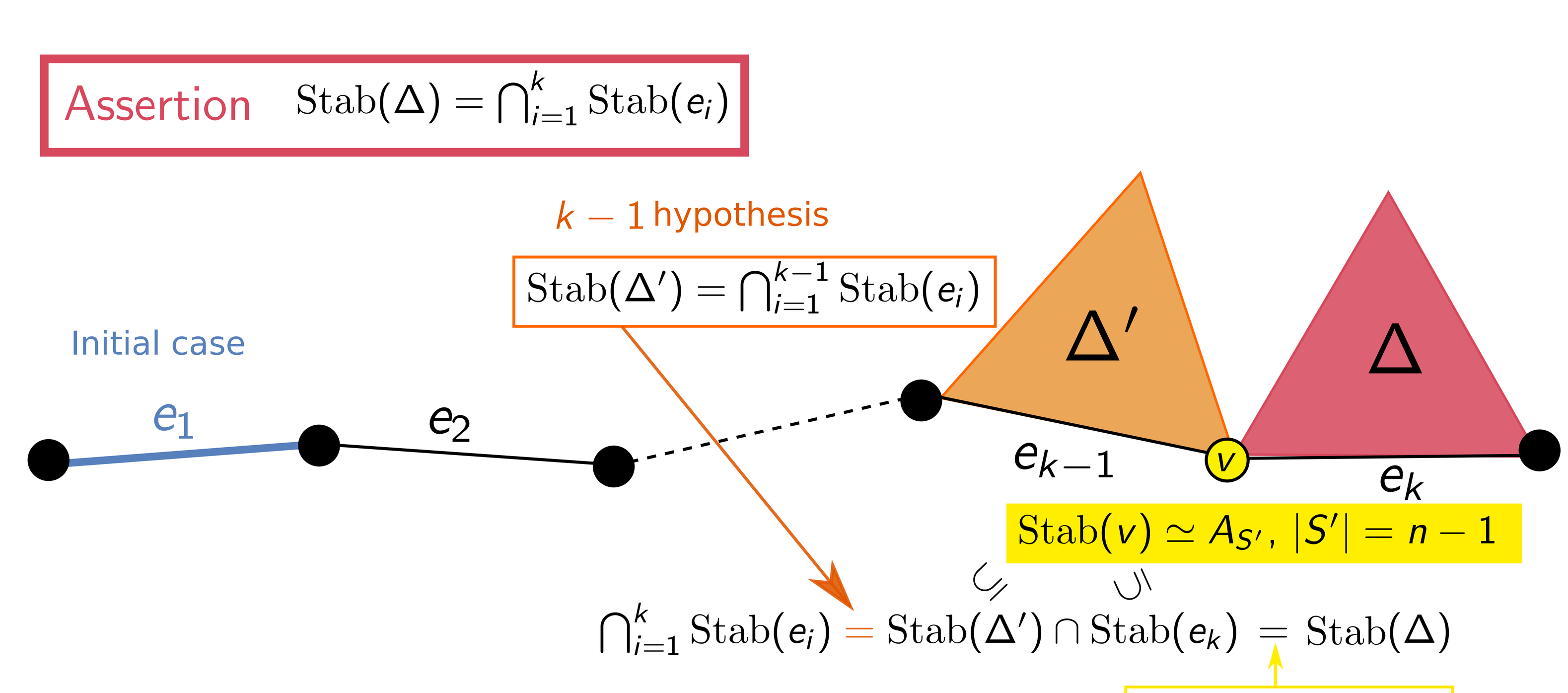
**Property 1** Parabolic subgroups of $A_S$ are stabilizers of simplices in $X_S$ and viceversa.




**Property 2** If $X_S$ is systolic, then any element of $A_S$ fixing two vertices of $X_S$ fixes every combinatorial geodesic between the vertices.

## Sketch of the proof of the theorem

1. By Prop 1, we need to say that the intersection the stabilizers of two simplices is the stabilizer of some simplex.
2. We select a geodesic between the simplices, by Prop 2, we need to study the intersection of the stabilizers of its edges.
3. We apply a double inductive argument (on the number of generators and on the length of the geodesic) using that the base case (2 generators) is spherical and it is already solved.




## Some references

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